Some properties of the value function and its level sets for affine control systems with quadratic cost

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Abstract

Let $T > 0$ fixed. We consider the optimal control problem for analytic affine systems : $\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)$, with a cost of the form : $C(u) = \int_0^T \sum_{i=1}^{m} u_i^2(t)dt$. For this kind of systems we prove that if there are no minimizing abnormal extremals then the value function S is subanalytic. Secondly we prove that if there exists an abnormal minimizer of corank 1 then the set of end-points of minimizers at cost fixed is tangent to a given hyperplane. We illustrate this situation in sub-Riemannian geometry.

Key words : optimal control, value function, abnormal minimizers, subanalyticity, sub-Riemannian geometry.

1 Introduction

Let M be an analytic Riemannian n-dimensional manifold and $x_0 \in M$. Consider the following *control system* :

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{1}$$

where $f = M \times \mathbb{R}^m \longrightarrow M$ is an analytic function, and the set of controls Ω is a subset of measurable mappings defined on $[0, T(u)]$ and taking their values in $\mathbb{R}^m$. The system is said to be *affine* if :

$$f(x, u) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \tag{2}$$

where the f_i's are analytic vector fields on M.

Let $T > 0$ fixed. We consider the *end-point mapping* $E : u \in \Omega \longmapsto x(T, x_0, u)$, where $x(t, x_0, u)$ is the solution of (1) associated to $u \in \Omega$ and starting from x_0 at $t = 0$. We endow the set of controls defined on $[0, T]$ with the L^2-norm topology. A trajectory $\widetilde{x}(t, x_0, \widetilde{u})$ denoted in short $\widetilde{x}$ is said to be *singular or abnormal* on $[0, T]$ if $\widetilde{u}$ is a singular point of the end-point mapping, i.e, the Fréchet derivative of E is not surjective at $\widetilde{u}$; otherwise it is said *regular*. We denote by $Acc(T)$ the set of end-points at $t = T$ of solutions of (1), u varying in Ω. The main problem of control theory is to study E and $Acc(T)$. Note that the latter is not bounded in general. In [18] one can find sufficient conditions so that $Acc(T)$ is compact, or has non-empty interior. Theorem 4.3 of this article states such a result for affine systems.

Consider now the following *optimal control problem* : among all trajectories of (1) steering 0 to $x \in Acc(T)$, find a trajectory minimizing the *cost function* : $C(u) = \displaystyle\int_0^T f^0(x_u(t), u(t))dt$, where f^0 is analytic. Such minimizers do not necessary exist ; the main argument to prove existence theorems is the lower semi-continuity of the cost function, see [12] or [18]. If $x \in Acc(T)$, we set $S(x) = \inf\{C(u) \ / \ E(u) = x\}$, otherwise $S(x) = +\infty$; S is called the *value function*. In general f^0 is choosen in such a way that the value function has a physical meaning : for instance the action in classical mechanics or in optics, the (sub)-Riemannian distance in (sub)-Riemannian geometry. We are interested in the regularity of the value function and the structure of its level sets. In (sub)-Riemannian geometry level sets of the distance are (sub)-Riemannian spheres. To describe these objects we need a category of sets which are stable under set operations and under proper analytic maps.

An important example of such a category is the one of *subanalytic sets* (see [13]). They have been utilized by several authors in order to construct an optimal synthesis or to describe $Acc(T)$ (see [11], [23]). Unfortunately this class is not wide enough : in [20], the authors exhibit examples of control systems in which neither S nor $Acc(T)$ are subanalytic. However Agrachev shows in [1] (see also [5], [16]) that if there are no abnormal minimizers then the sub-Riemannian distance is subanalytic in a pointed neighborhood of 0, and hence sub-Riemannian spheres of small radius are subanalytic. Following his ideas, we extend this result to affine control systems with quadratic cost (Theorem 4.4 and corollaries).

Abnormal minimizers are responsible for a phenomenon of *non-properness* (Proposition 5.3), which geometrically implies the following property : under certain assumptions the level sets of the value function are tangent to a given hyperplane at the end-point of the abnormal minimizer (Theorem 5.2). This result was first stated in [9] for sub-Riemannian systems to illustrate the Martinet situation.

An essential reasoning we will use in the proofs of these results is the following (see notably Lemma 4.8). We shall consider sequences of minimizing controls u_n associated to *projectivized Lagrange multipliers* $(p_n(T), p_n^0)$, so that

we have (see section 2) :

$$p_n(T)dE(u_n) = -p_n^0 u_n \tag{3}$$

Since (u_n) is bounded in L^2 we shall assume that u_n *converges weakly* to u. To pass to the limit in (3) we shall prove some regularity properties of the end-point mapping E (section 3). Contrarily to the sub-Riemannian case the strong topology on L^2 is not adapted in general for affine systems, whereas the weak topology gives nice compacity properties of the set of minimizing controls (see Theorem 4.12).

The outline of the paper is as follows : in section 2 we recall definitions of subanalytic sets and the Maximum Principle. In 3 we state some basic results on the regularity of the end-point mapping. Section 4 is devoted to continuity and subanalyticity of the value function S. Finally, in 5 the shape of the level sets of the value function in presence of abnormal minimizers is investigated. We illustrate this situation in sub-Riemannian geometry.

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2 Preliminaries

2.1 Subanalytic sets

Recall the following definitions, that can be found in [14], [15].

Definition 2.1. Let M be a finite dimensional real analytic manifold. A subset A of M is called *semi-analytic* iff, for every x in M, we can find a neighborhood U of x in M and $2pq$ real analytic functions g_{ij}, h_{ij} ($1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$) such that

$$A \cap U = \bigcup_{i=1}^{p} \{y \in U \;\; / \;\; g_{ij}(y) = 0 \;\text{ and }\; h_{ij}(y) > 0 \text{ for } j = 1 \dots q\}$$

We let $SEM(M)$ denote the family of semi-analytic subsets of M.

Unfortunately proper analytic images of semi-analytic sets are not in general semi-analytic. Hence this class must be extended :

Definition 2.2. A subset A of M is called *subanalytic* iff, for every x in M, we can find a neighborhood U of x in M and $2p$ pairs $(\phi_i^\delta, A_i^\delta)$ ($1 \leqslant i \leqslant p$ and $\delta = 1, 2$), where $A_i^\delta \in SEM(M_i^\delta)$ for some real analytic manifolds M_i^δ, and where the maps $\phi_i^\delta : M_i^\delta \to M$ are proper analytic, such that

$$A \cap U = \bigcup_{i=1}^{p} (\phi_i^1(A_i^1) \backslash \phi_i^2(A_i^2))$$

We let $SUB(M)$ denote the family of subanalytic subsets of M.

The class of subanalytic sets is closed under union, intersection, complement, inverse image of analytic maps, image of proper analytic maps. Moreover they are *stratifiable*. Recall the following :

Definition 2.3. Let M be a differentiable manifold. A stratum in M is a locally closed submanifold of M.

A locally-finite partition $\mathcal{S}$ of M is called a stratification of M if each S in $\mathcal{S}$ is a stratum such that :

$$\forall T \in \mathcal{S} \quad T \cap \mathrm{Fr}\ S \neq \emptyset \Rightarrow T \subset \mathrm{Fr}\ S \text{ and } \dim T < \dim \mathrm{S}$$

Endly, a map $f : M \to N$ between two manifolds is called *subanalytic* if its graph is a subanalytic set of $M \times N$.

The basic property of subanalytic functions which makes them useful in optimal control theory is the following. It can be found in [24].

Proposition 2.1. *Let M and N denote finite dimensional real analytic manifolds, and A be a subset of N. Given subanalytic maps $\phi : N \longrightarrow M$ and $f : N \longrightarrow I\!R$, we define :*

$$\forall x \in M \quad \psi(x) = \inf\{f(y) \ / \ y \in \phi^{-1}(x) \cap A\}$$

If $\phi_{/\bar{A}}$ is proper, then ψ is subanalytic.

2.2 Maximum principle and extremals

According to the *weak maximum principle* [21] the minimizing trajectories are among the singular trajectories of the end-point mapping of the *extended system* in $M \times I\!R$:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ \dot{x}^0(t) &= f^0(x(t), u(t)) \end{aligned} \tag{4}$$

They are called *extremals*. If E and C are differentiable, then there exists a *Lagrange multiplier* $(p(T), p^0)$ (defined up to a scalar) such that :

$$p(T)dE(u) = -p^0 dC(u) \tag{5}$$

where $dE(u)$ (resp. $dC(u)$) denotes the differential of E (resp. C) at u. Moreover, $(x(T), p(T))$ is the end-point of the solution of the following equations :

$$\dot{x} = \frac{\partial H}{\partial p} \ , \quad \dot{p} = -\frac{\partial H}{\partial x} \ , \quad \frac{\partial H}{\partial u} = 0 \tag{6}$$

where $H = < p, f(x,u) > + p^0 f^0(x,u)$ is the *Hamiltonian*, p is the adjoint vector, $<,>$ the inner product on M and p^0 is a constant. The abnormal trajectories correspond to the case $p^0 = 0$ and their role in the optimal control problem has to be analyzed. The extremals with $p^0 \neq 0$ are said normal. In this case p^0 is usually normalized to $-\frac{1}{2}$. We will use this normalization to prove Theorem 4.4. To prove Theorem 5.2 we will use another normalization by considering *projectivized Lagrange multipliers*, i.e. $(p(T), p^0) \in P(T^*M)$. We say that an extremal has *corank 1* if it has an unique projectivized Lagrange multiplier.

Affine systems Consider in particular analytic affine control systems on M :

$$\dot{x}(t) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \ \ x(0) = 0 \tag{7}$$

where the f_i's are analytic vector fields, with the problem of minimizing the following cost :

$$C(u) = \int_0^T \sum_{i=1}^{m} u_i^2(t) dt \tag{8}$$

The Hamiltonian is :

$$H(x, p, u) = < p, f_0(x) + \sum_{i=1}^{m} u_i f_i(x) > + p^0 \sum_{i=1}^{m} u_i^2$$

Parametrization of normal extremals We suppose $p^0 = -\frac{1}{2}$. Then normal controls can be computed from the equation : $\dfrac{\partial H}{\partial u} = 0$, and we get :

$$\forall i = 1 \dots m \quad u_i = < p, f_i(x) > \tag{9}$$

Putting in system (7), we get an analytic differential system in T^*M parametrized by the initial condition $p(0)$. From the general theory of ordinary differential equations we know that solutions depend *analytically* on their initial condition. Denote such a solution by $(x_{p(0)}, p_{p(0)})$. Let $u_{p(0)} = (< p_{p(0)}, f_1(x_{p(0)}) >, \dots, < p_{p(0)}, f_m(x_{p(0)}) >)$; from (9) $u_{p(0)}$ is a *normal control* associated to $x_{p(0)}$. Now we can define :

Definition 2.4. The mapping $\Phi \ : \ \begin{array}{ccc} T_{x_0}^* M & \longrightarrow & L^2([0,T], \mathbb{R}^m) \\ p(0) & \longmapsto & u_{p(0)} \end{array}$ is *analytic.*

This mapping will be useful to check subanalyticity of the value function in section 4.

3 Regularity of the end-point mapping

Let M be an analytic complete n-dimensional Riemannian manifold and $x_0 \in M$. Our point of view is local and we can assume : $M = \mathbb{R}^n, x_0 = 0$. We only consider *analytic affine control systems* (7). The statements in this section except for Proposition 3.7 are quite standard, and we include proofs only for convenience of the reader.

3.1 The end-point mapping

Let $T > 0$ and x_u be the solution, if exists, of the controlled system :

$$\dot{x}_u = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \ \ x_u(0) = 0$$

where $u = (u_1, \ldots, u_m) \in L^2([0,T], \mathbb{R}^m)$. Since we allow discontinuous controls, the meaning of solution of the previous differential system has to be clarified. In fact this means that the following integral equation holds :

$$\forall t \in [0,T] \quad x_u(t) = \int_0^T f_0(x_u(\tau)) + \sum_{i=1}^m u_i(\tau) f_i(x_u(\tau))\, d\tau$$

Definition 3.1. The *end-point mapping* is :

$$\begin{array}{rcl} E : \Omega & \longrightarrow & \mathbb{R}^n \\ u & \longmapsto & x_u(T) \end{array}$$

where $\Omega \subset L^2([0,T], \mathbb{R}^m)$ is the domain of E, that is the subset of controls u such that x_u is well-defined on $[0,T]$.

E is not defined on the whole L^2 because of *explosion phenomena*. For example consider the system $\dot{x} = x^2 + u$; then x_u is not defined on $[0,T]$ for $u = 1$ if $T \geqslant \frac{\pi}{2}$. Anyway we have the following :

Proposition 3.1. *Let $T > 0$ fixed. We consider the analytic control system (7). Then the domain Ω of E is open in $L^2([0,T], \mathbb{R}^m)$.*

Proof of the Proposition. It is enough to prove the following statement :

> If the trajectory x_u associated to u is well-defined on $[0,T]$, then the same is true for any control in a neighborhood of u in $L^2([0,T], \mathbb{R}^m)$.

Let V be a bounded open subset of $\mathbb{R}^n$ such that : $\forall t \in [0,T] \quad x_u(t) \in V$. Let $\theta \in C^\infty(\mathbb{R}^n, [0,1])$ with compact support K such that $\theta = 1$ on V. We can assume that $K = \bar{B}(0,R) = \{x \in \mathbb{R}^n \,/\, ||x|| \leqslant R\}$. For $i = 0 \ldots m$ we set $\tilde{f}_i = \theta f_i$. Then it is clear that x_u is also solution of : $\dot{x} = \tilde{f}_0(x) + \sum u_i \tilde{f}_i(x)$. For all $v \in L^2$ let $\tilde{x}_v$ be the solution of $\dot{\tilde{x}}_v = \tilde{f}_0(\tilde{x}_v) + \sum_{i=1}^m v_i \tilde{f}_i(\tilde{x}_v)$, $\tilde{x}_v(0) = 0$. We will prove that $\tilde{x}_v = x_v$ in a small enough neighborhood of u.

Lemma 3.2. *The $\tilde{f}_i$ are globally lipschitzian on $\mathbb{R}^n$, that is :*

$$\exists A > 0 \;/\; \forall i \in \{0, \ldots, m\} \;\; \forall y, z \in \mathbb{R}^n \quad ||\tilde{f}_i(y) - \tilde{f}_i(z)|| \leqslant A||y - z||$$

Proof of the Lemma. Let $i \in \{0, \ldots, m\}$. $\tilde{f}_i$ is C^1 hence is locally lipschitzian at any point :

$$\forall x \in \bar{B}(0,2R) \quad \exists \rho_x, A_x > 0 \;/\; \forall y, z \in B(x, \rho_x) \quad ||\tilde{f}_i(y) - \tilde{f}_i(z)|| \leqslant A_x||y - z||$$

From compacity, we can substract a finite number of balls which cover $\bar{B}(0,2R)$:

$$\exists p \in \mathbb{N} \;/\; \bar{B}(0,2R) \subset \bigcup_{j=1}^p B(x_j, \rho_{x_j})$$

Let $A = \sup_i A_{x_i}$ and $\rho = \frac{1}{2}\min(\frac{R}{2}, \min_i \rho_{x_i})$. Let us prove that $\tilde{f}_i$ is A-lipschitzian :
Let $y, z \in \mathbb{R}^n$.

1. if $||y-z|| \leqslant \rho$:
   - if $y,z \in \bar{B}(0,2R)$ then there exists $j \in \{1,\dots,p\}$ such that $y,z \in B(x_i,\rho_{x_i})$, and the conclusion holds.
   - if $y,z \notin \bar{B}(0,R)$ then $\tilde{f}_i(y)=\tilde{f}_i(z)=0$, and the inequality is still true.

   All other cases are impossible because $||y-z|| \leqslant \rho$.

2. if $||y-z|| > \rho$:
   Let $M = \sup\limits_{y,z\in K} ||\tilde{f}_i(y)-\tilde{f}_i(z)|| = \sup\limits_{y,z\in \mathbb{R}^n} ||\tilde{f}_i(y)-\tilde{f}_i(z)||$. Then :
   $$||\tilde{f}_i(y)-\tilde{f}_i(z)|| \leqslant M \leqslant \frac{M}{\rho}||y-z||$$
   and the conclusion holds if moreover A is choosen larger than $\frac{M}{\rho}$.

$\square$

For all $t\in[0,T]$ we have :

$$\begin{aligned}
||\tilde{x}_u(t)-\tilde{x}_v(t)|| = || &\int_0^t (\tilde{f}_0(\tilde{x}_u(\tau))-\tilde{f}_0(\tilde{x}_v(\tau)))d\tau \\
&+\int_0^t \sum_{i=1}^m v_i(\tau)(\tilde{f}_i(\tilde{x}_u(\tau))-\tilde{f}_i(\tilde{x}_v(\tau)))d\tau \\
&-\int_0^t \sum_{i=1}^m (v_i(\tau)-u_i(\tau))\tilde{f}_i(\tilde{x}_u(\tau))d\tau\, || \\
\leqslant A &\int_0^t (1+\sum_{i=1}^m |v_i(\tau)|)||\tilde{x}_u(\tau)-\tilde{x}_v(\tau)||d\tau + h_v(t)
\end{aligned}$$

where $h_v(t) = || \int_0^t \sum_{i=1}^m (v_i(\tau)-u_i(\tau))\tilde{f}_i(\tilde{x}_u(\tau))d\tau\, ||$.
Set $M' = \max\limits_i \sup\limits_{x\in\mathbb{R}^n} ||\tilde{f}_i(x)||$. We get from the *Cauchy-Schwarz inequality* :

$$\forall t\in[0,T] \quad h_v(t) \leqslant M'\sqrt{T}\,||v-u||_{L^2}$$

Hence for all $\varepsilon>0$ there exists a neighborhood U of u in L^2 such that :

$$\forall v\in U \quad \forall t\in[0,T] \quad h_v(t)\leqslant\varepsilon$$

Therefore :

$$\forall t\in[0,T] \quad ||\tilde{x}_u(t)-\tilde{x}_v(t)|| \leqslant A\int_0^t |(1+\sum_{i=1}^m v_i(\tau))|\,||\tilde{x}_u(\tau)-\tilde{x}_v(\tau)||d\tau+\varepsilon$$

We get from the *Gronwall Lemma* :

$$\forall t \in [0,T] \quad ||\tilde{x}_u(t) - \tilde{x}_v(t)|| \leqslant \varepsilon \mathrm{e}^{A\int_0^t |(1+\sum v_i(\tau))|d\tau} \leqslant \varepsilon \mathrm{e}^{AT+AK\sqrt{T}}$$

which proves that $(\tilde{x}_v)$ is uniformly close to $\tilde{x}_u = x_u$. In particular if the neighborhood U is small enough then : $\forall t \in [0,T] \quad x_v(t) \in V$, and hence $\tilde{x}_v = x_v$, which ends the proof. □

3.2 Continuity

If v and $v_n, n \in \mathbb{N}$ are elements of $L^2([0,T])$, we denote by $v_n \rightharpoonup v$ the weak convergence of the sequence (v_n) to v in L^2.

Proposition 3.3. *Let $u = (u_1, \ldots, u_m) \in \Omega$ and x_u be the solution of the affine control system :*

$$\dot{x}_u = f_0(x_u) + \sum_{i=1}^{m} u_i f_i(x_u), \;\; x_u(0) = 0$$

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2([0,T], \mathbb{R}^m)$. If $u_n \overset{L^2}{\rightharpoonup} u$ then for n large enough x_{u_n} is well-defined on $[0,T]$ and moreover $x_{u_n} \longrightarrow x_u$ uniformly on $[0,T]$.

Proof. The outline of the proof is the same as in Proposition 3.1. Let V be a bounded open subset of $\mathbb{R}^n$ such that : $\forall t \in [0,T] \quad x_u(t) \in V$. Let $\theta \in C^\infty(\mathbb{R}^n, [0,1])$ with compact support K such that $\theta = 1$ on V. We can assume that $K = \bar{B}(0,R) = \{x \in \mathbb{R}^n \,/\, ||x|| \leqslant R\}$. For $i = 0 \ldots m$ we set $\tilde{f}_i = \theta f_i$. Then it is clear that x_u is also solution of : $\dot{x} = \tilde{f}_0(x) + \sum u_i \tilde{f}_i(x)$. For all $n \in \mathbb{N}$ let $\tilde{x}_{u_n}$ be the solution of $\dot{\tilde{x}}_{u_n} = \tilde{f}_0(\tilde{x}_{u_n}) + \sum_{i=1}^{m} u_{n,i} \tilde{f}_i(\tilde{x}_{u_n})$, $\tilde{x}_{u_n}(0) = 0$. We will prove that if n is large enough then $\tilde{x}_{u_n} = x_{u_n}$.

For all $t \in [0,T]$ we have :

$$\begin{aligned} ||\tilde{x}_u(t) - \tilde{x}_{u_n}(t)|| = || & \int_0^t (\tilde{f}_0(\tilde{x}_u(\tau)) - \tilde{f}_0(\tilde{x}_{u_n}(\tau)))d\tau \\ & + \int_0^t \sum_{i=1}^{m} u_{n,i}(\tau)(\tilde{f}_i(\tilde{x}_u(\tau)) - \tilde{f}_i(\tilde{x}_{u_n}(\tau)))d\tau \\ & - \int_0^t \sum_{i=1}^{m} (u_{n,i}(\tau) - u_i(\tau))\tilde{f}_i(\tilde{x}_u(\tau))d\tau \, || \\ \leqslant A & \int_0^t (1 + \sum_{i=1}^{m} |u_{n,i}(\tau)|)||\tilde{x}_u(\tau) - \tilde{x}_{u_n}(\tau)||d\tau + h_n(t) \end{aligned}$$

where $h_n(t) = || \int_0^t \sum_{i=1}^{m} (u_{n,i}(\tau) - u_i(\tau))\tilde{f}_i(\tilde{x}_u(\tau))d\tau \, ||$. The aim is to make h_n uniformly small in t, and then to conclude we use the *Gronwall inequality.*

From the hypothesis : $u_n \rightharpoonup u$, we deduce : $\forall t \in [0,T] \quad h_n(t) \underset{n\to+\infty}{\longrightarrow} 0$. Let us prove that h_n tends uniformly to 0 as n tends to infinity. We need the following Lemma :

Lemma 3.4. *Let $a, b \in \mathbb{R}$ and E be a normed vector space. For all $n \in \mathbb{N}$ let $f_n : [a,b] \longrightarrow E$ be uniformly α-hlderian, that is :*

$$\exists \alpha, K > 0 \ / \ \forall n \in \mathbb{N} \ \forall x, y \in [a,b] \quad ||f_n(x) - f_n(y)|| \leqslant K||x-y||^\alpha$$

If the sequence (f_n) converges simply to an application f, then it tends uniformly to f.

Proof of the Lemma. Taking the limit as $n \longrightarrow \infty$, it is first clear that f is α-hlderian.

Let $\varepsilon > 0$ and $a = x_0 < x_1 < \cdots < x_p = b$ be a subdivision such that $\forall i \quad x_{i+1} - x_i < \frac{\varepsilon^{\frac{1}{\alpha}}}{2K}$. For all i, $f_n(x_i)$ tends to $f(x_i)$, hence :

$$\exists N \in \mathbb{N} \ / \ \forall n \geqslant N \ \forall i \in \{0, \ldots, p\} \quad ||f_n(x_i) - f(x_i)|| < \frac{\varepsilon}{3}$$

Let $x \in [a,b]$. Then there exists i such that $x \in [x_i, x_{i+1}]$. Hence :

$$\begin{aligned} ||f_n(x) - f(x)|| &\leqslant ||f_n(x) - f_n(x_i)|| + ||f_n(x_i) - f(x_i)|| + ||f(x_i) - f(x)|| \\ &\leqslant K||x - x_i||^\alpha + \frac{\varepsilon}{3} + K||x - x_i||^\alpha \\ &\leqslant \varepsilon \end{aligned}$$

□

Set : $M' = \max_i \sup_{x \in \mathbb{R}^n} ||\tilde{f}_i(x)||$. We get :

$$|h_n(x) - h_n(y)| \leqslant M' \left| \int_y^x \left(\sum_i |u_{n,i}(\tau)| + \sum_i |u_i(\tau)|\right) d\tau \right|$$

Moreover we get from the *Cauchy-Schwarz inequality* : $\displaystyle\int_y^x |u| \leqslant ||u||_{L^2} |x-y|^{\frac{1}{2}}$.

Furthermore the sequence (u_n) converges weakly, hence is bounded in L^2. Therefore there exists a constant K such that for all $n \in \mathbb{N}$:

$$|h_n(x) - h_n(y)| \leqslant K|x-y|^{\frac{1}{2}}$$

Hence from Lemma 3.4 we conclude that the sequence (h_n) tends uniformly to 0, that is :

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ / \ \forall n \geqslant N \ \forall t \in [0,T] \quad |h_n(t)| \leqslant \varepsilon$$

And hence, if $n \geqslant N$:

$$\forall t \in [0,T] \quad ||\tilde{x}_u(t) - \tilde{x}_{u_n}(t)|| \leqslant A \int_0^t |1 + \sum_{i=1}^m v_i(\tau)| \, ||\tilde{x}_u(\tau) - \tilde{x}_{u_n}(\tau)|| d\tau + \varepsilon$$

We get from the *Gronwall Lemma* :

$$\forall t \in [0,T] \quad ||\tilde{x}_u(t) - \tilde{x}_{u_n}(t)|| \leqslant \varepsilon \mathrm{e}^{AT+AK\sqrt{T}}$$

which proves that the sequence $(\tilde{x}_{u_n})$ tends uniformly to $\tilde{x}_u = x_u$. In particular if n is large enough then : $\forall t \in [0,T] \quad x_{u_n}(t) \in V$ and hence $\tilde{x}_{u_n} = x_{u_n}$, which ends the proof. □

Remark 3.1. This Proposition can be found in [22], but the author uses the following argument : if u_n tends weakly to 0 then $|u_n|$ tends weakly to 0, which is not true in general (take $u_n(t) = \cos nt$). That is the reason why we need Lemma 3.4. Otherwise the proof is the same as in [22].

To check differentiability in next subsection we will need the following result :

Proposition 3.5. *Let $u \in \Omega$ and x_u be the associated trajectory. Then for any bounded neighborhood U of u in $\Omega \subset L^2$ there exists a constant such that for all $v, w \in U$ and for all $t \in [0,T]$*

$$||x_v(t) - x_w(t)|| \leqslant C||v - w||_{L^2}$$

Proof. Writing :

$$\begin{aligned} \dot{x}_v &= f_0(x_v) + \sum_{i=1}^{m} v_i f_i(x_v) \\ \dot{x}_w &= f_0(x_w) + \sum_{i=1}^{m} w_i f_i(x_w) \end{aligned}$$

we get, for all $t \in [0,T]$:

$$\begin{aligned} ||x_v(t) - x_w(t)|| = || \int_0^t \Big(& \sum_i (v_i(s) - w_i(s)) f_i(x_v(s)) + f_0(x_v(s)) - f_0(x_w(s)) \\ & + \sum_i w_i(s)(f_i(x_v(s)) - f_i(x_w(s))) \Big) ds \, || \\ \leqslant \ & \sum_i \int_0^t |v_i - w_i| \, ||f_i(x_v)|| ds + \int_0^t ||f_0(x_v) - f_0(x_w)|| ds \\ & + \sum_i \int_0^t |w_i| \, ||f_i(x_v) - f_i(x_w)|| ds \end{aligned}$$

Now, if v and w are in a bounded neighborhood U of u in L^2, then according to Proposition 3.3, the trajectories x_v and x_w take their values in a compact K that depends only on U. The vector fields $f_0, f_1, \ldots, f_m$ being smooth, we claim that there exists a constant $M > 0$ such that for all $v, w \in U$ and for all i

$$\begin{aligned} ||f_i(x_v)|| &\leqslant M \\ ||f_i(x_v) - f_i(x_u)|| &\leqslant M||x_v - x_u|| \end{aligned}$$

Endly without loss of generality we can assume that U is contained in a ball of radius R centered in $O \in L^2$, so that

$$\forall w \in U \quad ||w||_{L^2} \leqslant R$$

Hence plugging in the upper inequality, and using the *Cauchy-Schwarz inequality*, we obtain :

$$\forall t \in [0,T] \quad ||x_v(t) - x_w(t)|| \leqslant A \int_0^t ||x_v(s) - x_w(s)|| ds + B||v - w||_{L^2}$$

where A and B are non negative constants. Finally we get from the *Gronwall Lemma* :

$$\forall t \in [0,T] \quad ||x_v(t) - x_w(t)|| \leqslant C||v - w||_{L^2}$$

with $C = B\mathrm{e}^{TA}$, which ends the proof. $\square$

3.3 Differentiability

Let $u \in \Omega$ and x_u the corresponding solution of the affine system (7). We consider the *linearized system* along x_u :

$$\dot{y}_v = A_u y_v + B_u v, \;\; y_v(0) = 0, \;\; v \in L^2 \tag{10}$$

where $A_u(t) = df_0(x_u) + \displaystyle\sum_{i=1}^{m} u_i df_i(x_u)$ and $B_u(t) = (f_1(x_u), \dots, f_m(x_u))$. Let M_u be the $n \times n$ matrix solution of

$$M_u' = A_u M_u, \;\; M_u(0) = Id \tag{11}$$

We have :

Proposition 3.6. *The end-point mapping* $E : \begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R}^n \\ u & \longmapsto & x_u(T) \end{array}$ *is L^2-Fréchet differentiable, and we have :*

$$\forall v \in \Omega \quad dE(u).v = \int_0^T M_u(T) M_u(s)^{-1} B_u(s) v(s) ds$$

Proof. Let $u \in L^2([0,T], \mathbb{R}^m)$. Let us prove that E is differentiable at u. Consider a neighborhood U of 0 in Ω, and let $v \in U$. Without loss of generality we can assume that there exists $R > 0$ such that for all $v \in U$: $||v||_{L^2} \leqslant R$. Let x_u (resp. x_{u+v}) the solution of the affine system (7) with the control u (resp. with the control $u + v$) :

$$\begin{aligned} \dot{x}_{u+v} &= f_0(x_{u+v}) + \sum_{i=1}^{m} (u_i + v_i) f_i(x_{u+v}) && (12) \\ \dot{x}_u &= f_0(x_u) + \sum_{i=1}^{m} u_i f_i(x_u) && (13) \end{aligned}$$

We get

$$\dot{x}_{u+v} - \dot{x}_u = \sum_{i=1}^m v_i f_i(x_{u+v}) + f_0(x_{u+v}) - f_0(x_u) + \sum_{i=1}^m u_i(f_i(x_{u+v}) - f_i(x_u))$$

Moreover, for all $i = 0 \dots m$:

$$\begin{aligned} f_i(x_{u+v}) - f_0(x_u) = df_i(x_u).(x_{u+v} - x_u) \ + \\ \int_0^1 (1-t) d^2 f_i(t x_u + (1-t) x_{u+v}).(x_{u+v} - x_u, x_{u+v} - x_u) dt \end{aligned}$$

Hence we obtain

$$\dot{\delta} = A_u \delta + B_u \delta + \gamma \tag{14}$$

where

$$\delta(t) = x_{u+v}(t) - x_u(t)$$

and

$$\begin{aligned} \gamma(t) = & \sum_{i=1}^m v_i(t) \int_0^1 df_i(s x_u + (1-s) x_{u+v}).(x_{u+v} - x_u) ds \\ & + \int_0^1 (1-t) d^2 f_0(s x_u + (1-s) x_{u+v}).(x_{u+v} - x_u, x_{u+v} - x_u) ds \\ & + \sum_{i=1}^m u_i(t) \int_0^1 (1-t) d^2 f_i(s x_u + (1-s) x_{u+v}).(x_{u+v} - x_u, x_{u+v} - x_u) ds \end{aligned}$$

Now for all $v \in U$ we have : $||v||_{L^2} \leqslant R$, thus from Proposition 3.5 there exists a compact K in $\mathbb{R}^n$ such that

$$\forall v \in U \ \ \forall s \in [0,1] \quad s x_u(s) + (1-s) x_{u+v}(s) \in K$$

The f_i's being smooth, we get, using again Proposition 3.5 :

$$\forall t \in [0,T] \quad ||\gamma(t)|| \leqslant c_1 ||v||_{L^2} \sum_{i=1}^m |v_i(t)| + c_2 ||v||_{L^2}^2 (1 + \sum_{i=1}^m |u_i(t)|)$$

Now solving equation (14) we obtain

$$\delta(t) = \int_0^t M_u(t) M_u(s)^{-1} B_u(s) v(s) ds + \int_0^t M_u(t) M_u(s)^{-1} \gamma(s) ds$$

Hence for $t = T$:

$$\begin{aligned} & ||x_{u+v}(T) - x_u(T) - \int_0^T M_u(T) M_u(s)^{-1} B_u(s) v(s) ds|| \\ & \qquad \leqslant c_1 ||v||_{L^2} \int_0^T \sum_{i=1}^m |v_i(t)| dt + c_2 ||v||_{L^2}^2 \int_0^T (1 + \sum_{i=1}^m |u_i(t)|) dt \\ & \qquad \leqslant c_3 ||v||_{L^2}^2 \end{aligned}$$

Moreover the mapping : $\begin{array}{ccc} L^2 & \longrightarrow & \mathbb{R}^n \\ v & \longmapsto & \int_0^T M_u(T)M_u(s)^{-1}B_u(s)v(s)ds \end{array}$ is linear and continuous. Hence the end-point mapping is Fréchet differentiable at u, and its differential at u is this latter mapping. □

Remark 3.2. Here it was proved that E is differentiable on L^2. It can be found also in [22]. Usually (see [21]) it is proved to be differentiable on L^∞.

Remark 3.3. The control u is abnormal and of corank 1 if and only if Im $dE(u)$ is an hyperplane of $\mathbb{R}^n$.

Proposition 3.7. *With the same assumptions as in Proposition 3.3, we have :*

$$u_n \overset{L^2}{\rightharpoonup} u \Rightarrow dE(u_n) \longrightarrow dE(u) \quad \textit{as } n \to +\infty$$

Proof. For $s \in [0,T]$, set $N_u(s) = M_u(T)M_u(s)^{-1}$.

Lemma 3.8. $N_u' = -N_uA_u, N_u(T) = Id$

Proof of the Lemma. The matrix N_uM_u is constant as t varies, hence $(N_uM_u)' = 0$. Moreover : $(N_uM_u)' = N_u'M_u + N_uA_uM_u$, and we get the Lemma. □

Lemma 3.9. $u_n \overset{L^2}{\rightharpoonup} u \Rightarrow N_{u_n} \longrightarrow N_u$ *uniformly on* $[0,T]$.

Proof of the Lemma. For $t \in [0,T]$, we have :

$$\begin{aligned} N_u(t) - N_{u_n}(t) = \int_0^t \Big(& N_{u_n}(s)\big(df_0(x_{u_n}(s)) + \sum_{i=1}^m u_{n,i}(s)df_i(x_{u_n}(s))\big) \\ & - N_u(s)\big(df_0(x_u(s)) + \sum_{i=1}^m u_i(s)df_i(x_u(s))\big)\Big)\, ds \\ = \int_0^t \Big(& \big(N_{u_n}(s) - N_u(s)\big)df_0(x_{u_n}(s)) \\ & + N_u(s)\big(df_0(x_{u_n}(s)) - df_0(x_u(s))\big) \\ & + \big(N_{u_n}(s) - N_u(s)\big)\sum_{i=1}^m u_{n,i}(s)df_i(x_{u_n}(s)) \\ & + N_u(s)\sum_{i=1}^m u_{n,i}(s)\big(df_i(x_{u_n}(s)) - df_i(x_u(s))\big) \\ & + N_u(s)\sum_{i=1}^m \big(u_{n,i}(s) - u_i(s)\big)df_i(x_u(s))\Big)\, ds \end{aligned}$$

From the hypothesis : $u_n \rightharpoonup u$, and from Proposition 3.3, we get that x_{u_n} tends uniformly to x_u, and hence for all i, $df_i(x_{u_n})$ tends uniformly to $df_i(x_u)$ on $[0,T]$.

Secondly, set : $h_n(t) = \int_0^t \sum_{i=1}^m \big(u_{n,i}(s) - u_i(s)\big) N_u(s) df_i(x_u(s)) ds$. Using the same argument as in the proof of Proposition 3.3, we prove that h_n tends uniformly to 0.

Hence we get the following inequality :

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \ / \ \forall n \geqslant N \quad \forall t \in [0,T]$$
$$||N_u(t) - N_{u_n}(t)|| \leqslant C \int_0^t ||N_u(s) - N_{u_n}(s)|| ds + \varepsilon$$

The *Gronwall inequality* gives us :

$$\forall t \in [0,T] \quad ||N_u(t) - N_{u_n}(t)|| \leqslant \varepsilon \mathrm{e}^{CT}$$

and the conclusion holds. □

Lemma 3.10. $u_n \overset{L^2}{\rightharpoonup} u \Rightarrow B_{u_n} \longrightarrow B_u$ *uniformly on* $[0,T]$.

Proof of the Lemma. From Proposition 3.3, we know that x_{u_n} tends uniformly to x_u, hence for all i, $f_i(x_{u_n})$ tends uniformly to $f_i(x_u)$, which proves the Lemma. □

We know that the differential of the end-point mapping has the following form :

$$\forall v \in L^2([0,T]) \quad dE(u).v = \int_0^T N_u(s) B_u(s) v(s) ds$$

Therefore from the preceeding Lemmas we get :

$$\forall v \in L^2([0,T]) \quad dE(u_n).v \longrightarrow dE(u).v$$

which ends the proof of the proposition. □

4 Properties of the value function and of its level sets

Let $T > 0$ fixed. Consider the affine control system (7) on $\mathbb{R}^n$ with cost (8). We denote by $Acc(T)$ the accessibility set in time T, that is the set of points that can be reached from 0 in time T.

4.1 Existence of optimal trajectories

The following result is a consequence of a general result from [18], p. 286.

Proposition 4.1. *Consider the analytic affine control system in* $\mathbb{R}^n$

$$\dot{x} = f_0(x) + \sum_{i=1}^m u_i f_i(x), \;\; x(0) = x_0, \;\; x(T) = x_1$$

with cost

$$C(u) = \int_0^T \sum_{i=1}^m u_i^2(t)dt$$

where $T > 0$ is fixed and the class Ω of admissible controllers is the subset of m-vector functions $u(t)$ in $L^2([0,T], I\!R^m)$ such that :

1. *$\forall u \in \Omega \quad x_u$ is well-defined on $[0,T]$.*
2. *$\exists B_T \ / \ \forall u \in \Omega \quad \forall t \in [0,T] \quad ||x_u(t)|| \leqslant B_T$.*

If there exists a control steering x_0 to x_1, then there exists an optimal control minimizing the cost steering x_0 to x_1.

4.2 Definition of the value function

Definition 4.1. Let $x \in \mathbb{R}^n$. Define $S : \mathbb{R}^n \longrightarrow \mathbb{R}^+ \cup \{+\infty\}$ by :

- If there is no trajectory steering 0 to x in time T, set : $S(x) = +\infty$.
- Otherwise set : $S(x) = \inf\{C(u) \ / \ u \in E^{-1}(x)\}$.

S is called the value function.

Definition 4.2. Let $r, T > 0$. Define the following level sets :

1. $M_r(T) = S^{-1}(r)$.
2. $M_{\leqslant r}(T) = S^{-1}([0,r])$.

Combining Proposition 4.1, arguments of Proposition 3.1 and the fact that the control $u = 0$, if admissible, is minimizing, we get :

Proposition 4.2. *Suppose the control $u = 0$ is admissible. Then there exists $r > 0$ such that any point of $M_{\leqslant r}(T)$ can be reached from 0 by an optimal trajectory.*

Hence if r is small enough, $M_r(T)$ (resp. $M_{\leqslant r}(T)$) is the set of extremities at time T of minimizing trajectories with cost equal to r (resp. lower or equal to r). It is a generalization of the (sub)-Riemannian sphere in (sub)-Riemannian geometry.

Theorem 4.3. *If r small enough then the subset $M_{\leqslant r}(T)$ is compact.*

Proof. First of all, with the same arguments as in Proposition 3.1, it is easy to see that $M_{\leqslant r}(T)$ is bounded if r is small enough. Now in order to prove that it is closed, consider a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $M_{\leqslant r}(T)$ converging to $x \in \mathbb{R}^n$. For each n let u_n be a minimizing control steering 0 to x_n in time T : $x_n = E(u_n)$ (the existence follows from Proposition 4.2). Then for all n, $C(u_n) \leqslant r$, which means that the sequence (u_n) is bounded in $L^2([0,T], \mathbb{R}^m)$, and therefore it admits a weakly converging subsequence. We can assume that $u_n \xrightarrow{L^2} u$. In particular : $C(u) \leqslant r$. Moreover from Proposition 3.3 we deduce : $x = E(u)$. Hence u is a control steering 0 to x in time T with a cost lower or equal to r. Thus : $x \in M_{\leqslant r}(T)$. This shows that the latter subset is closed. □

Remark 4.1. $M_r(T)$ is not necessarily closed. It is due to the fact that S can have discontinuities, see Example 4.2.

4.3 Regularity of the value function

We can now state the main theorem of this section.

Theorem 4.4. *Consider the analytic affine control system (7) with cost (8). Suppose r and T are small enough (so that any trajectory with cost lower than r is well-defined on $[0,T]$). Let K be a subanalytic compact subset of $M_{\leqslant r}(T)$. Suppose there is no abnormal minimizing geodesic steering 0 to any point of K. Then S is continuous and subanalytic on K.*

Corollary 4.5. *If r_0 and T are small enough and if there is no abnormal minimizer steering 0 to any point of $M_{\leqslant r_0}(T)$, then for any r lower than r_0, $M_r(T)$ and $M_{\leqslant r}(T)$ are subanalytic subsets of $\mathbb{R}^n$.*

This result generalizes to affine systems a result proved in [1] for sub-Riemannian systems (see also [5],[16]). The main argument to prove subanalyticity is the same as in [1], i.e. *the compactness of Lagrange multipliers associated to minimizers*, see Lemma 4.8 below.

If $\Omega = L^2([0,T],\mathbb{R}^m)$, i.e. if trajectories associated to any control u in L^2 are well-defined on $[0,T]$, then any point of $Acc(T)$ can be joined by a minimizing geodesic. Theorem 4.4 becomes :

Theorem 4.6. *If $\Omega = L^2([0,T],\mathbb{R}^m)$ and if there is no abnormal minimizing geodesic, then S is continuous on $\mathbb{R}^n$; moreover $Acc(T)$ is open and S is subanalytic on any subanalytic compact subset of $Acc(T)$.*

Proof of Corollary 4.5. If r_0 is small enough then from Theorem 4.3 $M_{\leqslant r_0}(T)$ is compact. We need a Lemma :

Lemma 4.7. *If $r < r_0$ then $M_{\leqslant r}(T)$ is contained in the interior of $M_{\leqslant r_0}(T)$.*

Proof of the Lemma. Let x be a point of $M_{\leqslant r}(T)$. From hypothesis, x is the extremity of a regular geodesic associated to a regular control u. Hence E is open in a neighborhood of u in L^2. Therefore there exists a neighborhood V of x such that any point of V can be reached by trajectories with cost close to r ; we can choose V so that their cost does not exceed r_0. Hence $V \subset M_{\leqslant r_0}(T)$, which proves that x belongs to the interior of $M_{\leqslant r_0}(T)$. $\square$

Let now K be a subanalytic compact subset containing $M_r(T)$ and $M_{\leqslant r}(T)$. We conclude using Theorem 4.4 and definition of the latter subsets. $\square$

We only prove Theorem 4.6. The proof of Theorem 4.4 is similar.

Proof of Theorem 4.6. First of all, note that $Acc(T)$ is open. For if $x \in Acc(T)$, let u be a minimizing control such that $x = E(u)$. From the assumption, u can not be abnormal. Thus it is normal, and $dE(u)$ is surjective. Hence from the *implicit function theorem*, E is open in a neighborhood of u. Therefore there exists a neighborhood of x contained in $Acc(T)$, thus the latter is open.

We first prove the continuity of S on $\mathbb{R}^n$. Take a sequence (x_n) of points of $\mathbb{R}^n$ converging to x. We shall prove that $S(x_n)$ converges to $S(x)$ by showing that $S(x)$ is the unique cluster point of the sequence $(S(x_n))$.

First case : $x \in Acc(T)$. Clearly : $Acc(T) = \underset{r\geqslant 0}{\cup} M_{\leqslant r}(T)$, and moreover : $r_1 < r_2 \Rightarrow M_{\leqslant r_1}(T) \subset M_{\leqslant r_2}(T)$. Hence there exists r such that x and x_n for n great enough are points of $M_{\leqslant r}(T)$. Now for each n there exists an optimal control u_n steering 0 to x_n, with a cost $C(u_n) = S(x_n) \leqslant r$. The sequence (u_n) is bounded in L^2, therefore it admits a weakly converging subsequence. We can assume that $u_n \rightharpoonup u$. From Proposition 3.3, we get : $x = E(u)$. Let a be a cluster point of $(S(x_n))_{n\in N}$. We can suppose that $S(x_n) \underset{n\to+\infty}{\longrightarrow} a$. From the weak convergence of u_n towards u we deduce that : $C(u) \leqslant a$. Therefore : $S(x) \leqslant a$. Let us prove that actually : $S(x) = a$. If not, then there exists a minimizing control v steering 0 to x with a cost b strictly lower than a. From hypothesis, v is normal, hence as before E is open in a (strong) neighborhood of v in L^2. It means that points near x can be attained with (not necessarily minimizing) controls with cost close to b. This contradicts the fact that $S(x_n)$ is close to a if n is large enough. Hence $a = S(x)$.

Second case : $x \notin Acc(T)$. Then $S(x) = +\infty$. Let us prove that $S(x_n) \to +\infty$. If not, considering a subsequence, we can assume that $S(x_n)$ converges to a. For each n let u_n be a minimizing control steering 0 to x_n. Again, the sequence (u_n) is bounded in L^2, hence we can assume that $u_n \rightharpoonup u \in L^2$. From the continuity of E we deduce : $x = E(u)$, which is absurd because x is not reachable. Hence : $S(x_n) \underset{n\to+\infty}{\longrightarrow} +\infty$.

Let us now prove the subanalyticity property. Let K a compact subset of $Acc(T)$. Here we use the first normalization for adjoint vectors (see section 2.2), that is we choose $p^0 = -\frac{1}{2}$ if the extremal is normal. The following Lemma asserts that the set of end-points at time T of the adjoint vectors associated to minimizers steering 0 to a point of K is bounded :

Lemma 4.8. *$\{p_u(T) \ / \ E(u) = x_u(T) \in K, u$ is minimizing$\}$ is a bounded subset of $\mathbb{R}^n$.*

Proof of Lemma 4.8. If not, there exists a sequence (x_n) of K such that the associated adjoint vector verifies : $||p_n(T)|| \underset{n\to+\infty}{\longrightarrow} +\infty$. Substracting a converging subsequence we can suppose that $x_n \underset{n\to+\infty}{\longrightarrow} x$. Now let u_n be a minimizing control associated to x_n, that is : $x_n = E(u_n)$. The vector $p_n(T)$ is a Lagrange

multiplier because u_n is minimizing, hence we have the following equality in L^2 :

$$p_n(T).dE(u_n) \stackrel{L^2}{=} -p^0 u_n$$

Dividing by $||p_n(T)||$ we obtain :

$$\frac{p_n(T)}{||p_n(T)||}.dE(u_n) \stackrel{L^2}{=} \frac{-p^0}{||p_n(T)||} u_n \tag{15}$$

Actually there exists r such that $K \subset M_{\leqslant r}(T)$. Hence : $C(u_n) \leqslant r$, and the sequence (u_n) is bounded in $L^2([0,T], \mathbb{R}^m)$. Therefore it admits a weakly convergent subsequence. We can assume that : $u_n \rightharpoonup u \in L^2$. Furthermore, the sequence $\left(\frac{p_n(T)}{||p_n(T)||}\right)$ is bounded in $\mathbb{R}^n$, hence up to a subsequence we have : $\frac{p_n(T)}{||p_n(T)||} \longrightarrow \psi \in \mathbb{R}^n$. Passing to the limit in (15), and using Proposition 3.7, we obtain :

$$\psi.dE(u) = 0 \quad \text{ where } x = E(u)$$

It means that u is an abnormal control steering 0 to x in time T. From the assumption, it is not minimizing, hence : $C(u) > S(x)$. On the one part, since u_n is minimizing, we get from the continuity of S that $C(u_n) \to S(x)$. On the other part, from the weak convergence of (u_n) towards u we deduce that $C(u) \leqslant S(x)$, and we get a contradiction. □

The previous Lemma asserts that end-points of adjoint vectors associated to minimizers reaching K are bounded. We now prove this fact for initial points of adjoint vectors :

Lemma 4.9. *$\{p_u(0) \ / \ E(u) \in K, u$ is minimizing$\}$ is a bounded subset of $\mathbb{R}^n$.*

Proof of Lemma 4.9. Let M_u be defined as in (11). From the classical theory we know that :

$$p_u(0) = p_u(T) M_u(T)$$

In the same way as in Lemma 3.9 we can prove :

$$u_n \stackrel{L^2}{\rightharpoonup} u \Longrightarrow M_{u_n}(T) \longrightarrow M_u(T) \quad \text{as } n \to +\infty$$

Now if the subset $\{p_u(0) \ / \ E(u) \in K, u \text{ is minimizing}\}$ were not bounded, there would exist a sequence (u_n) such that $||p_{u_n}(0)|| \longrightarrow +\infty$. Up to a subsequence we have : $u_n \rightharpoonup u, x_n = E(u_n) \longrightarrow x \in K$, and with the same arguments as in the previous Lemma, u is minimizing. Then it is clear that $||p_{u_n}(T)|| = ||p_{u_n}(0) M_{u_n}^{-1}(T)|| \underset{n \to +\infty}{\longrightarrow} +\infty$. This contradicts Lemma 4.8. □

Let now A be a subanalytic compact subset of $\mathbb{R}^n$ containing the bounded subset of Lemma 4.9. Then, if $x \in K$:

$$S(x) = \inf\{C o \phi(p) \ / \ p \in (E o \phi)^{-1}(x) \cap A\}$$

(see Definition 2.4 for Φ). Applying Proposition 2.1 we get the local subanalyticity of S. □

Remark 4.2. In sub-Riemannian geometry (i.e. $f_0 = 0$) the control $u = 0$ steers 0 to 0 with a cost equal to 0, thus is always a minimizing control. Moreover it is abnormal because Im $dE(0) = \text{Span}\ \{f_1(0), \dots, f_m(0)\}$ has corank $\geqslant 1$. Hence hypothesis of Corollary 4.5 *is never satisfied*. That is why the origin must be pointed out. In [1], Agrachev proves that the sub-Riemannian distance is subanalytic *in a pointed neighborhood of* 0, and hence that sub-Riemannian spheres of small radius are subanalytic.

The problem of subanalyticity of the sub-Riemannian distance at 0 is not obvious. Agrachev/Sarychev ([4]) or Jacquet ([16]) prove this fact under certain assumptions on the distribution. In fact for certain dimensions of the state space and codimensions of the distribution, the absence of abnormal minimizers (and hence subanalyticity of the spheres) and non-subanalyticity of the distance at 0 are both generic properties (see [3]).

Nevertheless for *affine systems* with $f_0 \neq 0$, the control $u = 0$ (which is always minimizing since $C(u) = 0$) is not in general abnormal. In fact it is not abnormal if and only if the linearized system along the trajectory of f_0 passing through 0 is controllable. Such conditions are well-known. For example we have the following :

> If $f_0(0) = 0$, set $A = df_0(0), B = (f_1(0), \dots, f_m(0))$. Then the control $u = 0$ is regular if and only if $rank(B|AB|\cdots|A^{n-1}B) = n$.

The regularity property is open, that is :

Proposition 4.10. *If u is regular then :*

$$\exists r > 0 \ / \ \forall v \quad ||u - v||_{L^2} \leqslant r \Rightarrow v \text{ is regular.}$$

Proof. If not : $\forall n \quad \exists v_n \ / \ ||u - v_n||_{L^2} \leqslant \frac{1}{n}$ and v_n is abnormal. Hence :

$$\exists p_n, ||p_n|| = 1 \ / \ \forall n \quad p_n.dE(v_n) = 0$$

Now the sequence (p_n) is bounded in $\mathbb{R}^n$, hence up to a subsequence p_n converges to $\psi \in \mathbb{R}^n$. On the other part, v_n converges to u in L^2, hence from Proposition 3.7 we get :

$$\psi.dE(u) = 0$$

which contradicts the regularity of u. □

Hence we can strengthen Corollary 4.5 and state :

Corollary 4.11. *Consider the affine system (7) with cost (8). If $u = 0$ is admissible on $[0, T]$ and is regular, then for any r small enough, S is continuous on $M_{\leqslant r}(T)$ and is subanalytic on any subanalytic compact subset of $M_{\leqslant r}(T)$. Moreover if r is small enough then $M_r(T)$ and $M_{\leqslant r}(T)$ are subanalytic subsets of $\mathbb{R}^n$.*

4.4 On the continuity of the value function

In Theorem 4.4 we proved in particular that if there is no abnormal minimizer then S is continuous on $M_{\leqslant r}(T)$. Otherwise it is wrong, as shown in the following example.

Example 4.1. Consider in $\mathbb{R}^2$ the affine system $\dot{x} = f_0(x) + uf_1(x)$ with

$$f_0 = \frac{\partial}{\partial x}, \quad f_1 = \frac{\partial}{\partial y}$$

Fix $T > 0$. It is clear that for any $u \in L^2$, x_u is well-defined on $[0, T]$. We have :

$$\begin{aligned} x(T) &= T \\ y(T) &= \int_0^T u(t)dt \end{aligned}$$

Hence

$$Acc(T) = \{(T, y) \ / \ y \in \mathbb{R}\}$$

The value function takes finite values in $Acc(T)$, and is infinite outside, thus is not continuous on $\mathbb{R}^n$. Note that for any control u, $dE(u)$ is never surjective, thus all trajectories are abnormal.

In the preceding example, S is however continuous in $Acc(T)$. But this is wrong in general, see the following example.

Example 4.2 (Working example). Consider in $\mathbb{R}^2$ the affine system $\dot{x} = f_0 + uf_1$ with

$$f_0 = (1 + y^2)\frac{\partial}{\partial x}, \quad f_1 = \frac{\partial}{\partial y}$$

Fix $T = 1$. The only abnormal trajectory γ is associated to $u = 0$: $\gamma(t) = (t, 0)$. Let $A = \gamma(1)$; we have $S(A) = 0$. The accessibility set at time 1 is :

$$Acc(1) = A \cup \{(x, y) \in \mathbb{R}^2 \ / \ x > 1\}$$

Consider now the problem of minimizing the cost $C(u) = \int_0^1 u^2(t)dt$. Normal extremals are solutions of :

$$\begin{array}{rclcrcl} \dot{x} & = & 1 + y^2 & \quad & \dot{y} & = & p_y \\ \dot{p}_x & = & 0 & & \dot{p}_y & = & -2yp_x \end{array}$$

Set $p_x = \lambda$. The area swept by $(x(1), y(1))$ as λ varies is represented on fig. 1.

The level sets $M_r(1)$ of the value function S are represented on fig. 2. The family $(M_r(1))_{r>0}$ is a partition of $Acc(1)$. Note that the slope of the vector u_r tends to infinity as r tends to 0.

The level sets $M_r(1)$ ramify at A, but do not contain this point, thus they are *not closed*. Now we can see that the value function S *is not continuous at A,*

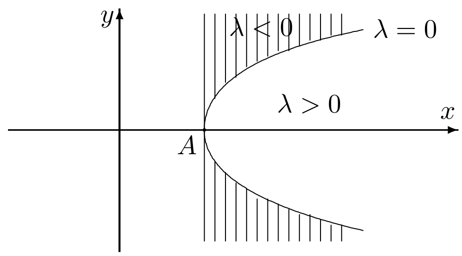


Figure 1: $\lambda < 0$

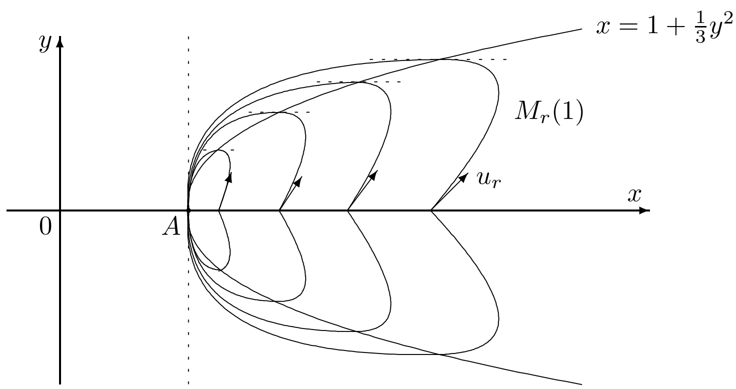


Figure 2: the level sets of the value function

even inside $Acc(1)$. Indeed on $M_r(1)$, S is equal to r, but at A we have $S(A) = 0$.

We can give an equivalent of the value function S near A in the area $(\lambda < 0)$ (see fig. 1). Computations lead to the following :

$$S(x, y) \sim \frac{1}{4}\frac{y^4}{x - 1}$$

Note that when $y \neq 0$ is fixed, if $x \to 1, x > 1$, then $\lambda \to -\infty$. This is a phenomenon of *non-properness* due to the existence of an abnormal minimizer. This fact was already encountered in sub-Riemannian geometry (see [9]). In the next section we explain this phenomenon.

In this example A is steered from 0 by the minimizing control $u = 0$. We

easily see that the set of minimizing controls steering 0 to points near A is not (strongly) compact in L^2. In fact we have the following :

Theorem 4.12. *Consider the analytic affine control system (7) with cost (8). Suppose r and T are small enough. Then S is continuous on $M_{\leqslant r}(T)$ if and only if the set of minimizing controls steering 0 to points of $M_{\leqslant r}(T)$ is compact in L^2.*

Remark 4.3. In sub-Riemannian geometry the value function S is always continuous, even though there may exist abnormal minimizers. This is due to the fact that S is the square of the sub-Riemannian distance (see for instance [6]). Note that in [16] (see also [1]) the set of minimizing controls joining $M_{\leqslant r}(T) = \overline{B}(0,r)$, r small enough, is proved to be compact in L^2.

Proof of Theorem 4.12. Suppose S is continuous on $M_{\leqslant r}(T)$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of minimizing controls steering O to points x_n of $M_{\leqslant r}(T)$. From Theorem 4.3 we can assume that x_n converges to $x \in M_{\leqslant r}(T)$. Let u be a minimizing control steering 0 to x. Since S is continuous we get : $||u_n||_{L^2} \underset{n\to+\infty}{\longrightarrow} ||u||_{L^2}$. The sequence (u_n) is bounded in L^2, hence up to a subsequence it converges weakly to $v \in L^2$ such that $||v||_{L^2} \leqslant ||u||_{L^2}$. On the other part from Proposition 3.3 we get $x = E(v)$. Therefore $||v||_{L^2} = ||u||_{L^2}$ since u is minimizing. Now combining the weak convergence of u_n to v and the convergence of $||u_n||_{L^2}$ to $||v||_{L^2}$ we get the (strong) convergence of u_n to v in L^2. This proves the compacity of minimizing controls since v is minimizing.

The converse is obvious. □

5 Role of abnormal minimizers

5.1 Theorem of tangency

This analysis is based on the sub-Riemannian Martinet case (see [9]) : it was shown that the exponential mapping is not proper and that in the generic case the sphere is tangent to the abnormal direction. This fact is general and we have the following results.

Lemma 5.1. *Consider the affine control system (7) with cost (8). Assume that there exists a minimizing geodesic γ on $[0,T]$ associated to an unique abnormal minimizing control u of corank 1, and that there exists $r > 0$ small enough such that $A = \gamma(T) \in M_{\leqslant r}(T)$. Denote by $(p_1, 0)$ the projectivized Lagrange multiplier at A. Let $\sigma(\tau)_{0<\tau\leqslant 1}$ be a curve on $M_{\leqslant r}(T)$ such that $\lim_{\tau\to 0} \sigma(\tau) = A$. For each τ we denote by $\mathcal{P}(\tau) \subset P(T^*_{\sigma(\tau)}M)$ the set of projectivized Lagrange multipliers at $\sigma(\tau)$: $\mathcal{P}(\tau) = \{(p_u(\tau), p^0_u) \ / \ E(u) = \sigma(\tau), u \text{ is minimizing}\}$. Then :*

$$\mathcal{P}(\tau) \underset{\tau\to 0}{\longrightarrow} \{(p_1, 0)\}$$

that is, each Lagrange multiplier of $\mathcal{P}(\tau)$ tends to $(p_1, 0)$ as $\tau \to 0$.

Proof. For each τ let u_τ a minimizing control steering 0 to $\sigma(\tau)$. For any $\tau \in]0,1]$ let $(p_\tau(T), p_\tau^0) \in \mathcal{P}(\tau)$. Let (ψ, ψ^0) be a cluster point at $\tau = 0$: there exists a sequence τ_n converging to 0 such that $(p_{\tau_n}(T), p_{\tau_n}^0) \longrightarrow (\psi, \psi^0)$. The sequence of controls (u_{τ_n}) is bounded in L^2, hence up to a subsequence it converges weakly to a control $v \in L^2$ such that $C(v) \leqslant r$. If r is small enough then from Proposition 4.2, v is admissible. Moreover from Proposition 3.3 we get : $E(v) = A$, and the assumption of the Lemma implies $v = u$. Now write the equality in L^2 defining the Lagrange multiplier :

$$p_{\tau_n}(T).dE(u_{\tau_n}) = -p_{\tau_n}^0 u_{\tau_n}$$

and passing to the limit we obtain (Proposition 3.7) :

$$\psi.dE(u) = -\psi^0 u$$

Since u has corank 1, we conclude : $(\psi, \psi^0) = (p_1, 0)$ in $P(T_A^*M)$. $\square$

Let $\widetilde{E}$ be the end-point mapping for the *extended system* in $\mathbb{R}^n \times \mathbb{R}$:

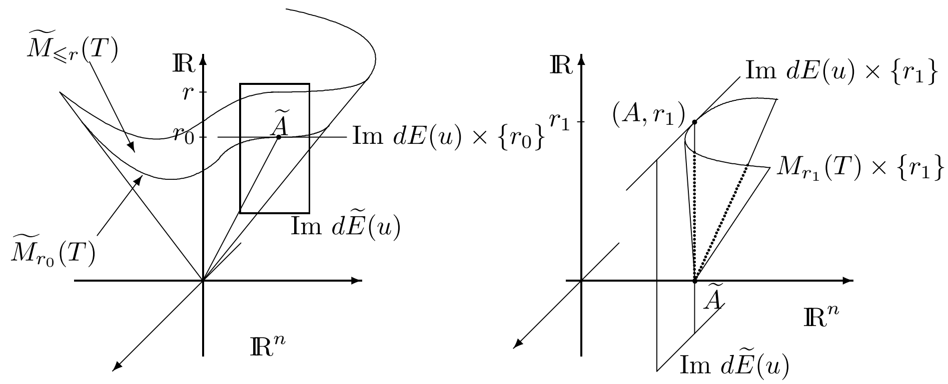
$$\begin{aligned} \dot{x} &= f_0(x) + \sum_{i=1}^m u_i f_i(x) \\ \dot{x}^0 &= \sum_{i=1}^m u_i^2 \end{aligned} \tag{16}$$

If $P \in M_r(T) \subset \mathbb{R}^n$, we denote by $\widetilde{P} = (P, r)$ the corresponding point in the augmented space. In the same way, we denote by $\widetilde{M}_r(T)$, $\widetilde{M}_{\leqslant r}(T)$ the corresponding sets in the augmented space.

Theorem 5.2. *Suppose the assumptions of Lemma 5.1 are fulfilled, and set $r_0 = S(A)$. If moreover $\widetilde{M}_{\leqslant r}(T)$ is C^1-stratifiable near $\widetilde{A} = (A, r_0)$, then the strata of $\widetilde{M}_{\leqslant r}(T)$ are tangent at $\widetilde{A}$ to the hyperplane Im $d\widetilde{E}(u)$ in $\mathbb{R}^n \times \mathbb{R}$. If moreover $A \in \overline{M_{r_1}(T)}$, $r_1 < r$, then $r_1 \geqslant r_0$ and the strata of $M_{r_1}(T)$ are tangent at A to the hyperplane Im $dE(u)$ in $\mathbb{R}^n$, see fig. 3.*

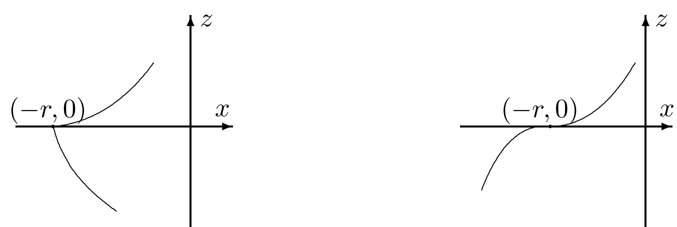
Proof. Let N be a stratum of $\widetilde{M}_{\leqslant r}(T)$ of maximal dimension near $\widetilde{A}$. Let $(\widetilde{\sigma}(\tau))_{0<\tau\leqslant 1}$ be a C^1 curve on $\overline{N}$ such that $\lim_{\tau\to 0} \widetilde{\sigma}(\tau) = \widetilde{A}$, and $\widetilde{\sigma}(\tau) = (\sigma(\tau), r_\tau)$. The aim is to prove that $\lim_{\tau\to 0} \widetilde{\sigma}'(\tau) \in \text{Im } d\widetilde{E}(u)$. From the assumption on the stratum $\widetilde{N}$, Im $d\widetilde{E}(u_\tau)$ is the tangent space to $\widetilde{N}$ at $\widetilde{\sigma}(\tau)$. By definition of the Lagrange multiplier, $(p_\tau(T), p_\tau^0)$ is normal to this subspace. Moreover $(p_1, 0)$ is normal to the hyperplane Im $d\widetilde{E}(u)$. Now from Lemma 5.1 we deduce : Im $d\widetilde{E}(u_\tau) \underset{\tau\to 0}{\longrightarrow}$ Im $d\widetilde{E}(u)$. The conclusion is now clear since $\widetilde{\sigma}'(\tau) \in \text{Im } d\widetilde{E}(u_\tau)$.

The second part of the Theorem is proved similarly. $\square$



sub-Riemannian case affine case

Figure 3: tangency in the augmented space



(a) flat case (b) generic case

Figure 4:

Example 5.1. In [9] a precise description of the SR sphere in 3-dimensional Martinet case is given. Generically, the abnormal minimizer has corank 1. The section of the sphere near the end-point of the abnormal minimizer with the plane $(y = 0)$ is represented on fig. 4, (b).

In the so-called flat case, the abnormal is not strict, and the shape of the sphere is represented on fig. 4, (a). In this case, the set of Lagrange multipliers associated to points near $(-r, 0)$ with $z < 0$ is bounded. That is why the slope does not converge to 0 as $z \to 0, z < 0$.

Hence Theorem 5.2 gives a geometric explanation to the pinching of the generic Martinet sphere near the abnormal direction.

Example 5.2. Consider again the affine system of Example 4.2. We proved that the set $M_r(1)$ is tangent at A to the hyperplane Im $dE(u) = \mathbb{R}\frac{\partial}{\partial y}$.

Note that, as in the preceding example, computations show that the branch

that ramifies at A is not subanalytic (see fig. 2). In fact, it belongs to the *exp-log category* (see [8]). More precisely this branch has the following graph near A :

$$x = 1 + F(y, \frac{\mathrm{e}^{-\frac{4r}{y^2}}}{y^3})$$

where F is a germ of analytic function at 0, and we have the following asymptotic expansion :

$$x = 1 + \frac{1}{4r}y^4 - 3y^2\mathrm{e}^{-\frac{4r}{y^2}} + \mathrm{o}(y^2\mathrm{e}^{-\frac{4r}{y^2}})$$

We get the following asymptotics of the value function :

$$S(x,y) = \frac{1}{4}\frac{y^4}{x-1} + \frac{y^4}{x-1}\mathrm{e}^{-\frac{y^2}{x-1}} + \mathrm{o}(\frac{y^4}{x-1}\mathrm{e}^{-\frac{y^2}{x-1}})$$

S is not subanalytic at A.

5.2 Interaction between abnormal and normal minimizers

Consider the affine system (7) with cost (8), and assume there exists a minimizing geodesic γ on $[0,T]$ associated to an unique abnormal control of corank 1. Denote by $A = \gamma(T)$.

Call *normal point* an end-point at time T of a normal minimizing geodesic. We make the following assumption :

> (H) For any neighborhood V of A there exists at least one normal point contained in $V \cap M_{\leqslant r}(T)$.

To describe the normal flow we use the first normalization of Lagrange multipliers (i.e. $p^0 = -\frac{1}{2}$ for normal extremals), which allows us to define the mapping Φ, see Definition 2.4. Now set $\exp = E \mathrm{o} \Phi$; it is a generalization of the (sub)-Riemannian exponential mapping. We have :

Proposition 5.3. *Under the preceding assumptions the mapping exp is not proper.*

Proof. Let (A_n) be a sequence of normal points of $M_{\leqslant r}(T)$ converging to A. For each A_n let $(p_n(T), -\frac{1}{2})$ be an associated Lagrange multiplier. Applying Lemma 5.1 we get : $\frac{p_n(T)}{\sqrt{||p_n(T)||^2+\frac{1}{4}}} \underset{n\to+\infty}{\longrightarrow} p_1$, $\frac{-\frac{1}{2}}{\sqrt{||p_n(T)||^2+\frac{1}{4}}} \underset{n\to+\infty}{\longrightarrow} 0$, thus in particular : $||p_n(T)|| \underset{n\to+\infty}{\longrightarrow} +\infty$. Now with the same arguments as in Lemma 4.9 we prove : $||p_n(0)|| \underset{n\to+\infty}{\longrightarrow} +\infty$. By definition : $A_n = \exp(p_n(0))$, hence exp is not proper. □

Remark 5.1. Conversely if exp is not proper then with the same arguments as in Lemma 4.8 there exists an abnormal minimizer. This shows the interaction between abnormal and normal minimizers. In a sense normal extremals recognize abnormal extremals. This phenomenon of non-properness is characteristic for abnormality.

5.3 Application : description of the sub-Riemannian sphere near an abnormal minimizer for rank 2 distributions

Let (M, Δ, g) be a sub-Riemannian structure of rank 2 on an analytic n-dimensional manifold M, $n \geqslant 3$, with an analytic metric g on Δ. Our point of view is local and we can assume that M is a neighborhood of $0 \in \mathbb{R}^n$, and that $\Delta = \text{Span}\,\{f_1, f_2\}$ where f_1, f_2 are independant analytic vector fields. Up to reparametrization, the problem of minimizing the cost (8) at time T fixed is equivalent to the time-optimal problem with the constraint $u_1^2 + u_2^2 \leqslant 1$. Let $\hat{\gamma}$ be a reference abnormal trajectory on $[0, r]$, associated to a control $\hat{u}$ and an adjoint vector $\hat{p}$. We suppose that $\hat{\gamma}$ is *injective*, and hence without loss of generality we can assume that $\hat{\gamma}(t) = \exp\, t f_1(0)$.

We make the following assumptions :

(H_1) Let $K(t) = \text{Im}\, dE_t(\hat{u}) = \text{Span}\,\{ad^k f_1.f_{2|\hat{\gamma}}, k \geqslant 0\}$ be the first Pontriaguine cone along $\hat{\gamma}$. We assume that $K(t)$ has codimension 1 for any $t \in]0, r]$ and is spanned by the $n-1$ first vectors $ad^k f_1.f_{2|\hat{\gamma}}, k = 0 \dots n-2$.

(H_2) $ad^2 f_2.f_{1|\hat{\gamma}} \notin K(t)$ along $\hat{\gamma}$.

(H_3) $f_{1|\hat{\gamma}} \notin \{ad^k f_1.f_{2|\hat{\gamma}}, k = 0 \dots, n-3\}$.

Under these assumptions $\hat{\gamma}$ has corank 1. Moreover from [19] $\hat{\gamma}$ is minimizing if r is small enough, and $\hat{u}$ is the unique minimizing abnormal control steering 0 to $\hat{\gamma}(r)$. Hence assumptions of Lemma 5.1 are fulfilled.

Let now V be a neighborhood of $\hat{p}(0)$ such that all abnormal geodesics starting from 0 with $p_\gamma(0) \in V$ satisfy also the assumptions $(H_1) - (H_2) - (H_3)$. Note that if V is small enough, they are also injective. We have, see [4] and [19] :

Proposition 5.4. *There exists $r > 0$ such that the previous abnormal geodesics are optimal if $t \leqslant r$.*

Corollary 5.5. *The end-points of these abnormal minimizers form in the neighborhood of $\hat{\gamma}(r)$ an analytic submanifold of dimension $n-3$ if $n \geqslant 3$, reduced to a point if $n = 3$, contained in the sub-Riemannian sphere $S(0, r)$.*

Hence in the neighborhood of $\hat{\gamma}(r)$ the sub-Riemannian sphere $S(0, r)$ splits into two parts : the *abnormal part* and the *normal part*. To describe $S(0, r)$ near $\hat{\gamma}(r)$ we have to *glue* them together. If the hypothesis of C^1-stratification is fulfilled then the normal part ramifies tangently to the abnormal part in the sense of Theorem 5.2. This gives us a qualitative description of the sphere near $\hat{\gamma}(r)$.

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